Minimum-Mass Design of a Plate-Like Structure for Specified Fundamental Frequency

JEAN-LOUIS ARMAND*
Stanford University, Stanford, Calif.

The purpose of this paper is to present a new approach to the problem of the minimum-mass design of two-dimensional continuous structures which are required to satisfy a constraint of a dynamic or aeroelastic nature, expressed in the form of one or more partial differential equations. The minimization of a functional subject to constraints of this form belongs to a wider class of problems, encountered in the theory of optimal control of systems with distributed parameters. Classical methods of optimal control theory are here extended to two dimensions in order to derive the set of necessary conditions for an extremum. They are then applied to the theoretical case of the minimum-mass design of a simply-supported shear plate for a given fundamental frequency of vibration under the structural mass assumption. An equation for the optimal displacement, which is the expression of a general optimality criterion in structural design due to Prager and rendering the necessary conditions also sufficient, is derived and solved uniquely in closed form for any shape of the plate. Results are presented for a square and rectangular shape, and for a circular plate. The case of an inequality constraint applied to the thickness (minimum thickness) is also examined.

Introduction

THE problem of the minimum-mass design of structures, with constraints imposed on a free-vibration frequency or an aeroelastic eigenvalue, has attracted considerable attention recently.† Several papers published in the last three years describe different approaches for either continuous or discrete structures. Among those techniques the use of optimal control theory, suggested by Ashley and McIntosh[1] and later applied by their students,[2,3] proved to be very powerful in the case of fairly simple continuous systems.

The structures considered up to now have been one-dimensional; that is, the imposed constraints are expressed in the form of ordinary differential equations in one independent spatial variable. A beam problem is obviously one-dimensional, and so is a circular plate when the constraint is on a frequency of vibration whose corresponding mode is axially symmetrical.[4]

Problems of structural optimization in which the constraints are in the form of partial differential equations have hardly been investigated heretofore. The only tentative efforts known to the author are those of Johnson[5] and Haug,[6] who investigated independently the problem of the optimization of a plate for a fixed first frequency of vibration. Both make use of the method of discretization of the structure; unfortunately, it is therefore impossible to give an estimate of precision which would guarantee the validity of their optimal solution. Moreover, Johnson's square plate is divided into 25 elements, the symmetry of the problem further reducing the accuracy to only 6 elements. Haug, on the other hand, presents an adaptation of the powerful method of steepest descent, developed in the numerical solution of control problems and applied to a two-dimensional system. This approach seems, by its generality, very promising and should lead to the numerical solution of numerous interesting examples.

The purpose of this paper is to extend the optimal control theory technique to a simple two-dimensional structural optimization problem, the minimum-mass design of a simply-supported rectangular shear plate with a prescribed fundamental frequency of free vibration. A closed-form expression for the thickness distribution is found, and uniqueness of the optimal solution proven by using an extremely general theorem due to Prager.

1. The Shear Plate: Definition and Analysis

A shear plate is a plate-like structure such that the bending rigidity for normal loads is negligible. The running load is thus borne by shear only. For free vibrations, the uniform plate is analogous to a membrane under constant tension. The plate is referred to a set of rectangular axes Ox, Oy in the middle plane. It occupies a simply connected domain D with a piecewise smooth boundary ∂D.

The reference structure is assumed to be made up of two parts with drastically different structural properties: a constant fraction δ_2 is nonstructural, while the remaining part of the mass, labeled structural, is originally in the proportion δ_1. H being a reference thickness having the dimension of a length, the thickness at each point is of the form $Hh(x,y)$, $h(x,y)$ being a nondimensional quantity function of the coordinates of the point and expressed as:

$$h(x,y) = \delta_1 h^*(x,y) + \delta_2 \tag{1}$$

where

$$\delta_1 + \delta_2 = 1 \tag{2}$$

Note that the non-structural mass is not under the control of

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* Graduate Student, Department of Aeronautics and Astronautics; now Associate Professor of Applied Mechanics, Federal University of Rio de Janeiro. Associate Member AIAA.

† A full session at the ASME/AIAA 10th Structures, Structural Dynamics and Materials Conference held in New Orleans in April 1969 was devoted to the subject.

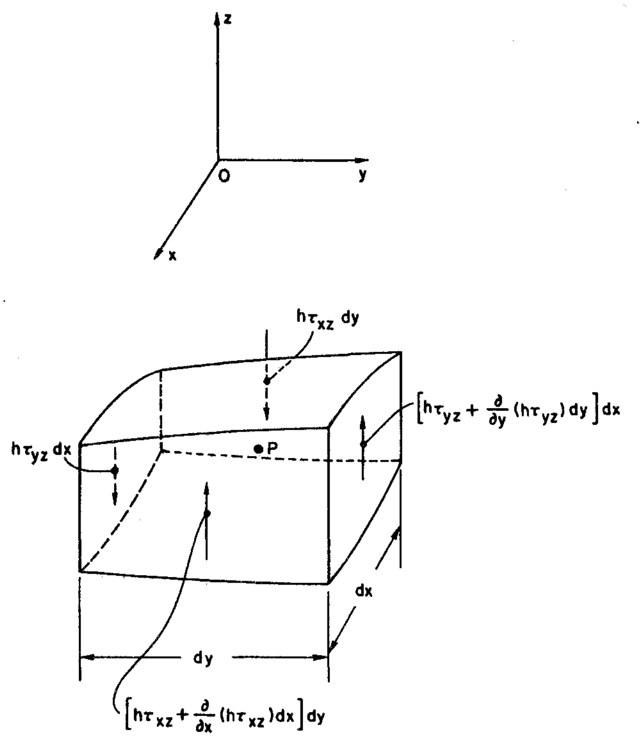

Fig. 1 Shear forces acting on the sides of an infinitesimal element of plate.

the designer. The object is to minimize the total mass by acting on the virtual thickness h^* which is effective in determining shear rigidity.

The consideration of the forces acting upon an infinitesimal element of the plate, with sides parallel to the coordinate axes (Fig. 1), leads to the equilibrium equation:

$$(\partial/\partial x)(h^*\tau_{xz}) + (\partial/\partial y)(h^*\tau_{yz}) + q/H = 0 \qquad (3)$$

where q is the intensity of the continuously distributed normal load. The classical stress-strain relations link the shearing stress components acting on the sides normal to Ox and Oy, τ_{xz} and τ_{yz}, respectively, to the corresponding shearing strain components, γ_{xz} and γ_{yz}, themselves related to the normal displacement w, as follows:

$$\begin{aligned} \tau_{xz} &= G\gamma_{xz} = G\partial w/\partial x \\ \tau_{yz} &= G\gamma_{yz} = G\partial w/\partial y \end{aligned} \qquad (4)$$

G being the modulus of elasticity in shear.

The equilibrium equation is then rewritten:

$$(\partial/\partial x)(h^*\partial w/\partial x) + (\partial/\partial y)(h^*\partial w/\partial y) + q/GH = 0 \qquad (5)$$

Free vibrations of the shear plate are governed by the equation

$$(\partial/\partial x)(h^*\partial w/\partial x) + (\partial/\partial y)(h^*\partial w/\partial y) - (\rho/G)(\delta_1 h^* + \delta_2)\partial^2 w/\partial t^2 = 0 \qquad (6)$$

in which ρ represents the density of the material of the plate. For a simply-supported rectangular plate, with uniform thickness H and sides of length a and b, respectively, the fundamental frequency of vibration is found to be

$$\omega_f = \pi(G/\rho)^{1/2}[(1/a^2) + (1/b^2)]^{1/2} \qquad (7)$$

For a simply-supported circular plate with uniform thickness and radius a, the fundamental frequency is

$$\omega_f = 2.4048(G/\rho a^2)^{1/2} \qquad (8)$$

corresponding to an axisymmetrical mode.

In order to find the optimal thickness distribution of the simply-supported plate of minimum weight occupying a domain D and having a given fundamental frequency of vibration ω_f, one has to minimize the surface integral:

$$J = \iint_D h^*(x,y)dxdy \qquad (9)$$

with the partial differential equation constraint

$$\partial/\partial x(h^*\partial w/\partial x) + \partial/\partial y(h^*\partial w/\partial y) + (\rho/G)\,\omega_f^2(\delta_1 h^* + \delta_2)w = 0 \qquad (10a)$$

and the boundary condition

$$w = 0 \quad \text{along} \quad \partial D \qquad (10b)$$

2. Necessary Conditions for the Stationary Value of a Functional under Constraints Expressed as Partial Differential Equations

A problem of the type above is encountered in the theory of optimal control of systems with distributed parameters, which extends the classical optimal control theory to situations where the system is distributed over not only time but some spatial domain as well. This is a very new and broad field of investigation.[7] Let us derive the necessary conditions for an extremal, for a broad class of problems as stated below.

In a closed planar domain D, with a piecewise smooth boundary ∂D represented in parametric form relative to a set of rectangular coordinates (x,y) by the equations

$$\begin{aligned} x &= \alpha(\sigma) \\ y &= \beta(\sigma) \end{aligned} \qquad (11)$$

where α and β are continuous, piecewise differentiable functions of the parameter σ, one considers a system of first-order partial differential equations of the form:

$$\begin{aligned} \partial z_i/\partial x &= X_i(\mathbf{z},\mathbf{u};x,y) \\ \partial z_i/\partial y &= Y_i(\mathbf{z},\mathbf{u};x,y) \qquad i = 1,2,\ldots,n \end{aligned} \qquad (12a)$$

It is also assumed that the first $m \leq n$ functions z_i are prescribed along portions Σ of ∂D:

$$z_i|_\Sigma = z_i(\sigma), \qquad i = 1,2,\ldots,m \qquad (12b)$$

For a system governed by a set of equations in the form (12a), the vector function $\mathbf{z} = (z_1,z_2,\ldots,z_n)$ of the arguments x,y describes the mechanical system itself and the z_i play the role of state variables. The vector function $\mathbf{u} = (u_1,u_2,\ldots,u_p)$ of the same arguments represents the distributed controls.

The optimization problem is now stated as follows: in a suitable class of functions determine the state variables $\mathbf{z}$ and control variables $\mathbf{u}$ so as to minimize the functional

$$J = \int_{\partial D} \ell(\mathbf{z};\sigma)d\sigma + \iint_D L(\mathbf{z},\mathbf{u};x,y)dxdy \qquad (13)$$

where ℓ and L are given continuous and at least once differentiable functions of each of their arguments, subject to side conditions (12a) and (12b).

The above is a problem of the Mayer-Bolza type in two dimensions. It may be complicated by considering boundary controls on movable boundaries and/or constraints on the state or control variables. A treatment of this optimization problem, for a broad class of constraints, is found in Ref. 8. Our approach to the simplest problem already described will, however, be different.

Following the methods of optimal control theory in one single variable,[9] we adjoin the system of partial differential Eqs. 12a to J with the help of vector multiplier functions

$\lambda(x,y)$ and $\mu(x,y)$.

$$J = \int_{\partial D} \ell(\mathbf{z};\sigma)d\sigma + \iint_D \{L(\mathbf{z},\mathbf{u};x,y) + \lambda^T(x,y) \times [\mathbf{X}(\mathbf{z},\mathbf{u};x,y) - \partial\mathbf{z}/\partial x] + \mu^T(x,y)[\mathbf{Y}(\mathbf{z},\mathbf{u};x,y) - \partial\mathbf{z}/\partial y]\} \times dxdy \quad (14)$$

We define a scalar function, the Hamiltonian, as

$$H(\mathbf{z},\mathbf{u};x,y) = L(\mathbf{z},\mathbf{u};x,y) + \lambda^T(x,y)\mathbf{X}(\mathbf{z},\mathbf{u};x,y) + \mu^T(x,y)\mathbf{Y}(\mathbf{z},\mathbf{u};x,y) \quad (15)$$

J is rewritten then as:

$$J = \int_{\partial D} \ell(\mathbf{z};\sigma)d\sigma + \iint_D \{H(\mathbf{z},\mathbf{u};x,y) - \lambda^T(x,y)(\partial\mathbf{z}/\partial x) - \mu^T(x,y)(\partial\mathbf{z}/\partial y)\}dxdy$$

We now add and subtract the quantity:

$$[(\partial\lambda^T/\partial x) + (\partial\mu^T/\partial y)]\mathbf{z}$$

to the second integrand. J takes the form:

$$J = \int_{\partial D} \ell(\mathbf{z};\sigma)d\sigma + \iint_D \left\{H(\mathbf{z},\mathbf{u};x,y) + \left(\frac{\partial\lambda^T}{\partial x} + \frac{\partial\mu^T}{\partial y}\right)\mathbf{z}\right\} dxdy - \iint_D \{(\partial/\partial x)(\lambda^T\mathbf{z}) + (\partial/\partial y)(\mu^T z)\}dxdy$$

Now, by the use of Green's formula,‡ the third integral becomes a line integral as follows:

$$\iint_D \{(\partial/\partial x)(\lambda^T\mathbf{z}) + (\partial/\partial y)(\mu^T\mathbf{z})\}dxdy = \int_{\partial D} (\mu^T dx - \lambda^T dy)\mathbf{z} \quad (16)$$

Then the integral (16) takes the form:

$$\int_{D\partial} [\mu^T\alpha'(\sigma) - \lambda^T\beta'(\sigma)]\mathbf{z}d\sigma \quad (17)$$

and J becomes:

$$J = \int_{\partial D} \{\ell(\mathbf{z};\sigma) - [\lambda^T\beta'(\sigma) - \mu^T\alpha'(\sigma)]\mathbf{z}\}d\sigma + \iint_D \left\{H(\mathbf{z},\mathbf{u};x,y) + \left(\frac{\partial\lambda^T}{\partial x} + \frac{\partial\mu^T}{\partial y}\right)\mathbf{z}\right\} dxdy \quad (18)$$

Now consider the variation in J due to variations in the control vector $\mathbf{u}(x,y)$:

$$\delta J = \int_{\partial D} [\partial\ell/\partial\mathbf{z} + \lambda^T\beta'(\sigma) - \mu^T\alpha'(\sigma)]\delta\mathbf{z}d\sigma + \iint_D \{(\partial H/\partial\mathbf{z} + \partial\lambda^T/\partial x + \partial\mu^T/\partial y)\delta\mathbf{z} + (\partial H/\partial\mathbf{u})\delta\mathbf{u}\}dxdy \quad (19)$$

We now choose the multiplier functions $\lambda(x,y)$ and $\mu(x,y)$ to cause the coefficients of $\delta\mathbf{z}$ in the previous surface integral to vanish:

$$\partial\lambda^T/\partial x + \partial\mu^T/\partial y = -(\partial H/\partial\mathbf{z}) \quad (20a)$$

Along the portions of ∂D where the z_i are prescribed

$$\delta z_i = 0, \qquad i = 1,2,\ldots,m$$

Along the other portions and for the z_j, $j = m + 1, \ldots, n$ which are not at all prescribed, we ask the relations

$$\lambda^T\beta'(\sigma) - \mu^T\alpha'(\sigma) = -(\partial\ell/\partial\mathbf{z}) \quad (20b)$$

to hold along ∂D: they serve as boundary conditions for the system (20a).

Equation (19) then becomes:

$$\delta J = \iint_D (\partial H/\partial\mathbf{u})\delta\mathbf{u}dxdy$$

‡ It is the counterpart in two dimensions of the classical integration by parts in one dimension.

For an extremum, δJ must be zero for arbitrary $\delta\mathbf{u}(x,y)$; this can only happen if:

$$\partial H/\partial\mathbf{u} = 0 \quad (21)$$

Equations (21) are analogous to the control equations derived in one-dimensional optimal control theory.

Equations (20a, 20b and 21) are the Euler-Lagrange equations of the classical calculus of variations, for two independent variables. They form a set of necessary conditions for an optimum.

In summary, to find a control vector function $\mathbf{u}(x,y)$ that produces a stationary value of the performance index:

$$J = \int_{\partial D} \ell(\mathbf{z};\sigma)d\sigma + \iint_D L(\mathbf{z},\mathbf{u};x,y)dxdy \quad (13)$$

we must solve the following system of partial differential equations in D:

$$\partial\mathbf{z}/\partial x = \mathbf{X}(\mathbf{z},\mathbf{u};x,y)$$

$$\partial\mathbf{z}/\partial y = \mathbf{Y}(\mathbf{z},\mathbf{u};x,y) \quad (12a)$$

$$\partial\lambda/\partial x + \partial\mu/\partial y = -(\partial H/\partial\mathbf{z})^T \quad (20a)$$

$$\partial H/\partial\mathbf{u} = 0 \quad (21)$$

where the Hamiltonian H is defined as:

$$H(\mathbf{z},\mathbf{u};x,y) = L(\mathbf{z},\mathbf{u};x,y) + \lambda^T(x,y)\mathbf{X} + \mu^T(x,y)\mathbf{Y} \quad (15)$$

The boundary conditions are provided by Eq. (12b), together with the transversality conditions

$$\lambda_i\beta'(\sigma) - \mu_i\alpha'(\sigma) = -\partial\ell/\partial z_i \quad (20b)$$

which hold along the portions of ∂D where z_i is not prescribed.

3. Application to the Shear-Plate

For the problem at hand, the constraint (10) is broken down into a system of first-order partial differential equations in the form (12a), as follows§

$$\partial z_1/\partial x = z_2/h^* \qquad \partial z_1/\partial y = z_3/h^* \qquad \partial z_2/\partial x = u_1$$

$$\partial z_2/\partial y = u_2 \qquad \partial z_3/\partial x = u_3$$

$$\partial z_3/\partial y = -u_1 - k^2(h^* + \delta_2/\delta_1)z_1 \quad (22a)$$

Here w has been renamed z_1, and the constant k^2 is defined as

$$k^2 = (\rho/G)\omega_f{}^2\delta_1$$

The boundary condition is

$$z_1 = 0 \qquad \text{on } \partial D \quad (22b)$$

The Hamiltonian is as follows

$$H = h^* + \lambda_1 z_2/h + \lambda_2 u_1 + \lambda_3 u_3 + \mu_1 z_3/h + \mu_2 u_2 + \mu_3[-u_1 - k^2(h^* + \delta_2/\delta_1)z_1] \quad (23)$$

and the necessary conditions are

$$\partial\lambda_1/\partial x + \partial\mu_1/\partial y = k^2\mu_3(h^* + \delta_2/\delta_1)$$

$$\partial\lambda_2/\partial x + \partial\mu_2/\partial y = -(\lambda_1/h^*)$$

$$\partial\lambda_3/\partial x + \partial\mu_3/\partial y = -(\mu_1/h^*)$$

$$1 - (\lambda_1 z_2/h^{*2}) - (\mu_1 z_3/h^{*2}) - k^2\mu_3 z_1 = 0$$

$$\lambda_2 - \mu_3 = 0, \quad \mu_2 = 0, \quad \lambda_3 = 0 \quad (24a)$$

§ Any partial differential equation of order superior to one or any system of such equations may be written in the form (12), with the number of dependent variables increased as necessary. A general method to achieve such a decomposition, by no means unique, is given in Ref. 10.

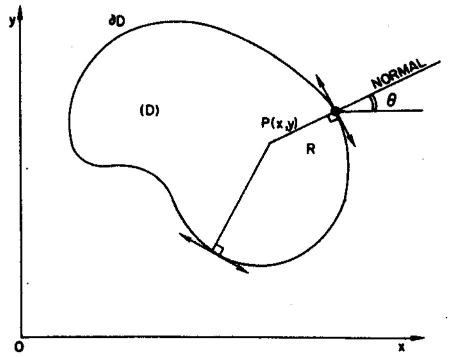

Fig. 2 Solution of the partial differential Eq. (26) for a general domain *D*: definition and interpretation of the different quantities introduced.

The boundary conditions read

$$\begin{aligned} z_1 &= 0 \\ \lambda_2\beta'(\sigma) - \mu_2\alpha'(\sigma) &= 0 \\ \lambda_3\beta'(\sigma) - \mu_3\alpha'(\sigma) &= 0 \text{ on } \partial D \end{aligned} \tag{24b}$$

and reduce to, in light of Eq. (24a)

$$\begin{aligned} z_1 &= 0 \\ \lambda_2 = \mu_3 &= 0 \quad \text{on } \partial D \end{aligned} \tag{24c}$$

The first three equations of the system (24a) are very similar to the first two and the last of the system (22a). Previous experience in one-dimensional cases[2] leads to the assumptions

$$\lambda_1 = z_2/c^2, \qquad \mu_1 = z_3/c^2, \qquad \lambda_2 = \mu_3 = -(z_1/c^2) \tag{25}$$

where c^2 is any constant.
These are compatible with the boundary conditions

$$z_1 = \lambda_2 = \mu_3 = 0 \qquad \text{on } \partial D$$

w is then found to be the solution of the nonlinear, first-order partial differential equation¶

$$(\partial w^2/\partial x) + (\partial w^2/\partial y) = c^2 + k^2w^2 \tag{26a}$$

together with the boundary condition

$$w = 0 \qquad \text{on } \partial D \tag{26b}$$

The form of Eq. (26a) suggests the introduction of the auxiliary function $\theta(x,y)$ defined by

$$\begin{aligned} \partial w/\partial x &= (c^2 + k^2w^2)^{1/2} \cos\theta(x,y) \\ \partial w/\partial y &= (c^2 + k^2w^2)^{1/2} \sin\theta(x,y) \end{aligned} \tag{27}$$

The compatibility conditions require θ to be a solution of the linear first-order partial differential equation

$$\cos\theta \frac{\partial\theta}{\partial x} + \sin\theta \frac{\partial\theta}{\partial y} = 0 \tag{28}$$

The characteristics, given by the differential system

$$dx/\cos\theta = dy/\sin\theta = d\theta/0$$

are straight lines. On the boundary, w has the constant value 0; therefore

$$dw \equiv (\partial w/\partial x)dx + (\partial w/\partial y)dy = 0$$

or

$$\cos\theta dx + \sin\theta dy = 0$$

This shows that θ is nothing but the angle that the normal to the contour ∂D, oriented outwards, makes with the x axis. The characteristics are therefore the normals to this contour (Fig. 2). At any point P interior to the domain

$$\begin{aligned} dw &= (c^2 + k^2w^2)^{1/2}(\cos\theta dx + \sin\theta dy) \\ &= (c^2 + k^2w^2)^{1/2}dR \end{aligned}$$

R being the distance from P to the boundary ∂D. The solution to Eqs. (26a) and (26b) is then simply

$$w = \frac{\epsilon c}{k} \sinh(kR), \qquad \epsilon = \pm 1 \tag{29}$$

For a given contour, there are usually many normals that can be drawn from a point interior to the domain it encloses (Fig. 2). Thus the question arises as to which normal must be considered to define the distance R. It seems logical, and turns out to be the only solution for a rectangular domain, to take for R the shortest distance from P to the boundary ∂D.

Equation (26a) expresses the conservation of the so-called Lagrangian density which was pointed out for the optimal frequency design of one-dimensional structures by Ashley and McIntosh[1] and, for various design criteria by Prager and Taylor,[11] who applied it to sandwich structures. Prager[12,13] proved that, for a very general class of structural optimization problems, of which the optimal frequency design of a plate with prescribed vanishing displacement along the edge is a particular case,[10] this conservation condition is a sufficient one for an extremum. Equation (26) is therefore necessary and sufficient, and the optimal displacement given by Eq. (29) is therefore found to be unique. The corresponding thickness distribution is uniquely found in the following sections for the case of a rectangular and a circular shape, respectively, thus proving the uniqueness of the solution to the optimization problem.

4. Optimization of a Rectangular Shear Plate

The aforementioned result is valid for any domain D under the general assumptions outlined previously. In the case of a rectangular plate with sides of length a and b ($a \geq b$), the optimal displacement has a different analytical expression in each of the four regions defined in Fig. 3, whose boundaries are the loci of the points from where two normals of equal length can be drawn to the contour. The computation of the optimal thickness distribution is straightforward and re-

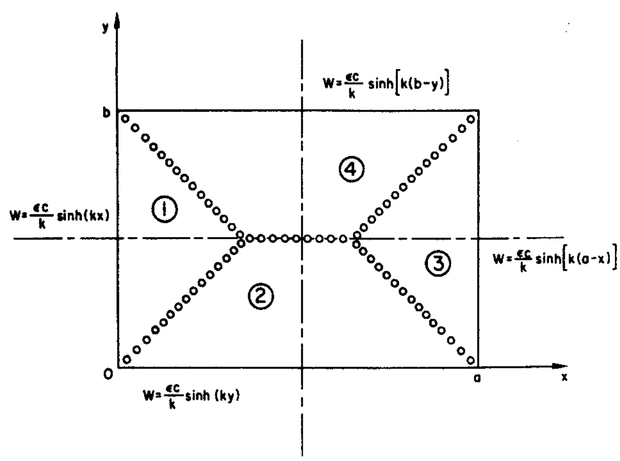

Fig. 3 The division of the rectangular plate into four regions, and the corresponding expressions for the optimal displacement *w*.

¶ A very ingenious change of dependent variable, reducing Eq. (26a) to the equation $p^2 + q^2 = 1$ encountered in optics and also in the theory of plastic torsion of a cylindrical beam, was suggested by E. O. A. Naumann, Convair Div. of General Dynamics, whose correspondence is gratefully acknowledged.

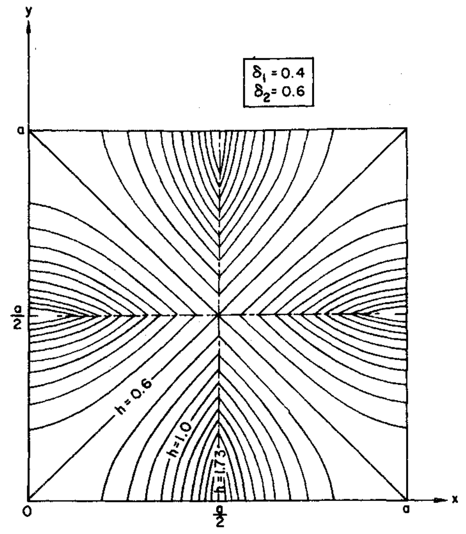

Fig. 4 Optimal configuration of a square simply-supported shear plate.

quires solving an ordinary differential equation in each of the above domains: in Eq. (1)

$$h(x,y) = (\delta_2/2) \times \{1 + [\cosh^2(ky)/\cosh^2(kx)]\} \quad (30)$$

in the portion of Eq. (2) such that $0 \leq x \leq b/2$ and $a - b/2 \leq x \leq a$:

$$h(x,y) = (\delta_2/2) \times \{1 + \{[\cosh^2(kx)/\cosh^2(ky)]\} \quad (31)$$

in the remaining portion of Eq. (2) such that $b/2 \leq x \leq a - b/2$:

$$h(x,y) = (\delta_2/2) \times \{1 + [\cosh^2(kb/2)/\cosh^2(ky)]\} \quad (32)$$

The expression for h in Eq. (3) is obtained from that in Eq. (1) by changing x into $a - x$; in Eq. (4), from that in Eq. (2), by changing y into $b - y$.

The optimal mass ratio, defined as the ratio of the mass of the optimal structure to that of a uniform structure with constant thickness H of the same frequency, is found to be

$$M = \delta_2 \left\{\frac{1}{2} + \frac{2ab}{\pi^2\delta_1(a^2+b^2)} \sinh^2 \left\{\frac{\pi}{2a}[\delta_1(a^2+b^2)]^{1/2}\right\} + \frac{a-b}{2\pi\delta_1[(a^2+b^2)]^{1/2}} \sinh\left\{\frac{\pi}{a}[\delta_1(a^2+b^2)]^{1/2}\right\}\right. \quad (33)$$

The contour lines (lines of constant thickness), corresponding

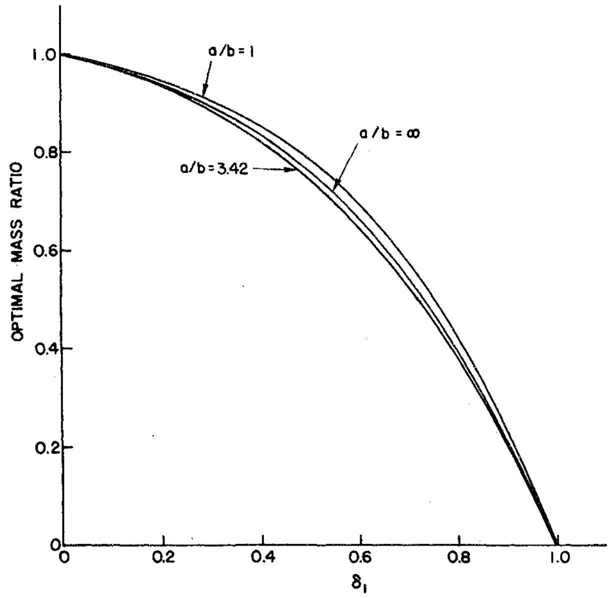

Fig. 6 Variation of the optimal mass ratio with δ_1 for different values of the ratio a/b.

to the case where 40% of the mass is initially structural and thus allowed to vary, are presented in Figs. 4 and 5. These are drawn, respectively, for a square plate and for a rectangular plate with $a/b = 1.5$. The optimal non-dimensional thickness distribution h attains its minimum of 0.6, corresponding to $h^* = 0$, along the lines already described in Fig. 3. A thickness of 1.73 is attained at the four points middle of the sides in the case of the square plate. The over-all mass reductions relative to $h^* = 1$ are found to be 14.3% and 15.9%, respectively; this is an encouraging result in view of the 60% of the initial mass which is not under the control of the designer. The variation of the optimal mass ratio with δ_1 for different values of the ratio a/b is represented in Fig. 6.

The optimal distribution found above is unfortunately not of any practical value, as "hinges" will develop along those lines where h^* is found to vanish. It is therefore very desirable to impose a constraint h_0 on the virtual thickness h^*, which is equivalent to imposing a minimum actual thickness $h_0 = \delta_1 h_0^* + \delta_2$.

The general theory of optimization outlined above is easily extended when constraints on the control variables taking

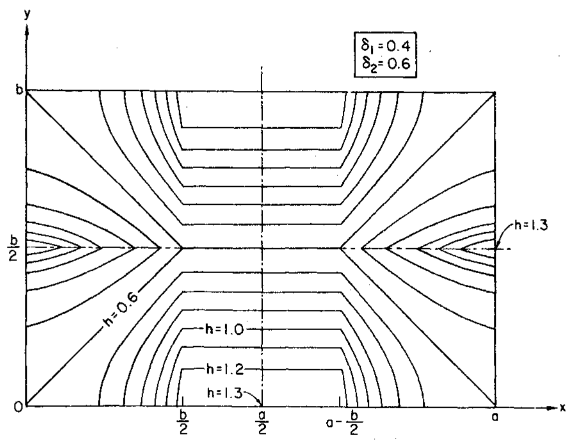

Fig. 5 Optimal configuration of a rectangular simply-supported shear plate ($a/b = 1.5$).

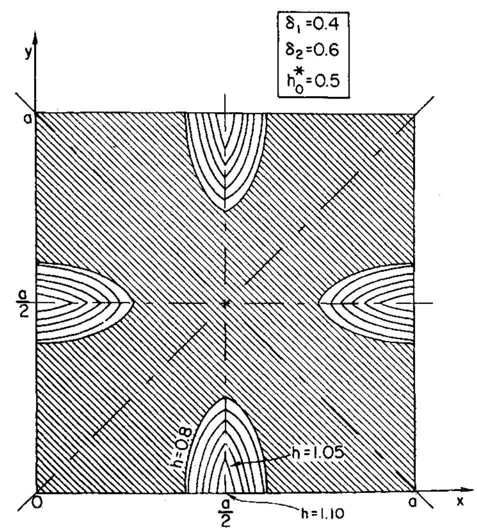

Fig. 7 Optimal thickness distribution for a simply-supported square shear plate with a minimum-thickness constraint.

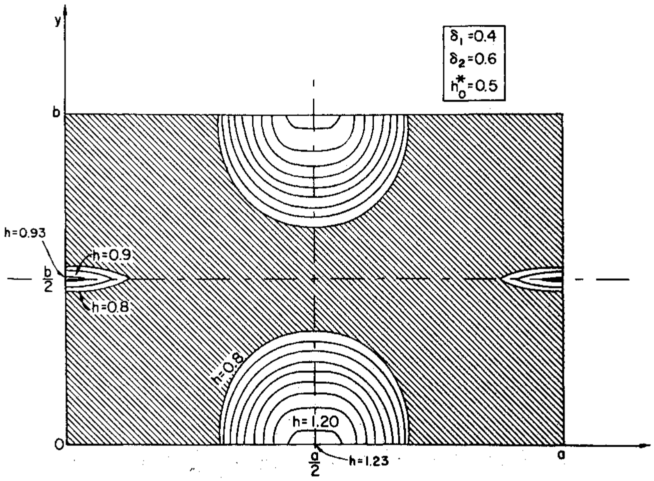

Fig. 8 Optimal thickness distribution for a simply-supported rectangular shear plate with a minimum-thickness constraint (a/b = 1.5).

the form of finite inequalities are present.[10] The optimal configuration found for the two same structures above, in the case of a virtual thickness constraint of 0.5 representing an actual constraint of 0.8 on the total thickness h are represented in Figs. 7 and 8. In the case of the square plate, 84.4% of the plate surface is at the minimum allowed thickness of 0.8. For the rectangular plate, the minimum allowed thickness is reached on 77.6% of the total area. The advantages of imposing a constraint are quite obvious, especially when one considers that the mass savings are only slightly inferior (14.1% and 15.5%, respectively) and that such optimal structures come close to practical attainment in cases where fixed frequency might constitute a meaningful design criterion.

5. Optimization of a Circular Shear Plate

We now turn our attention to the optimization of a circular simply-supported shear plate. The optimal thickness distribution being assumed to be rotation-invariant, i.e., its expression in polar coordinates to be independent of θ, the problem might also, as pointed out in the introduction, be viewed as a one-dimensional one, and solved using the classical methods of optimal control theory, which will provide two ways of finding the solution.

In polar coordinates (r,θ), an axisymmetric optimal solution h^*†† has to satisfy Eq. (10) rewritten as:

$$(h^*w')' + (1/r)h^*w' + k^2(h^* + \delta_2/\delta_1)w = 0 \qquad (34)$$

where (′) denotes the derivative taken with respect to r.

With the value of the optimal displacement given by Eq. (29) where R is taken equal to $a - r$,

$$w = \frac{\epsilon c}{k}\sinh[k(a-r)] \qquad (35)$$

the optimal virtual thickness h^* has to satisfy the ordinary differential equation:

$$h^{*\prime} + \left\{\frac{1}{r} - 2k\tanh[k(a-r)]\right\}h^* - k\frac{\delta_2}{\delta_1}\times \tanh[k(a-r)] = 0 \qquad (36)$$

For the same reasons than for the rectangular plate (continuity of the shear), the optimal virtual thickness h^* has to vanish at the center of the plate. The solution is thus found to be, after some manipulation:

$$h^* = \frac{\delta_2}{\delta_1}\,\frac{\sinh(2ka) - \sinh[2k(a-r)] - 2kr\cosh[2k(a-r)]}{8kr\cosh^2[k(a-r)]} \qquad (37)$$

and the optimal thickness distribution is given by:

$$h = \delta_2\left\{\frac{1}{2} + \frac{1}{4\cosh^2[k(a-r)]} + \frac{\sinh(2ka) - \sinh[2k(a-r)]}{8kr\cosh^2[k(a-r)]}\right\} \qquad (38)$$

We recall that:

$$k = \omega_f[(\rho/G)\delta_1]^{1/2} = 2.4048[(\delta_1)^{1/2}/a]$$

The optimal mass ratio is then equal to:

$$M = \frac{\int_0^a 2\pi hr dr}{\pi a^2}$$

†† The superscript (*) has the same significance as before, the actual thickness being $h = \delta_1 h^* + \delta_2$, with $\delta_1 + \delta_2 = 1$.

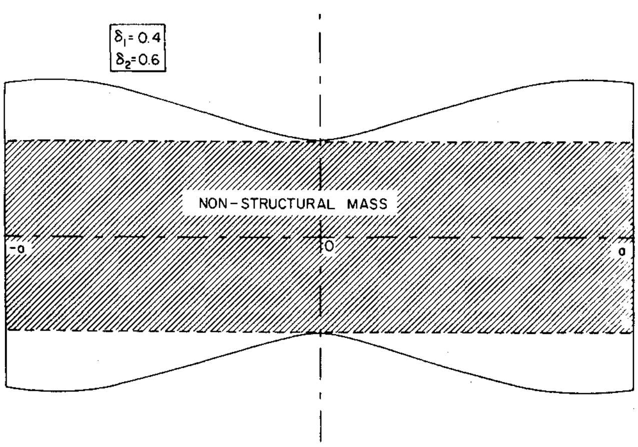

Fig. 9 Diametral cross section of an optimal circular simply-supported shear plate for which 40% of the mass is structural.

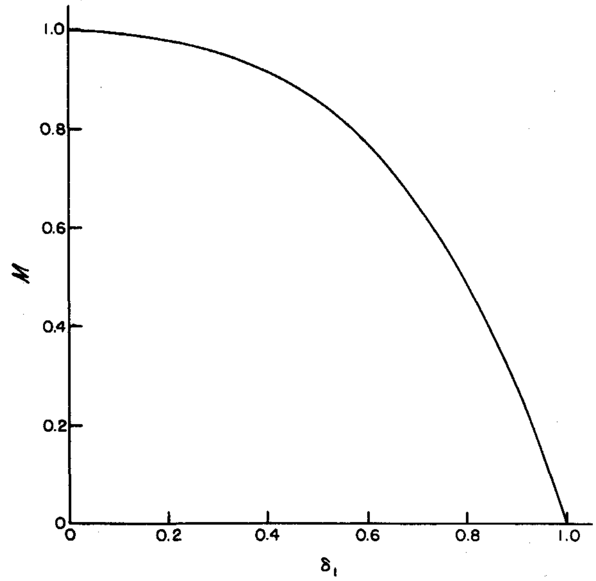

Fig. 10 Variation of the optimal mass ratio with the proportion of structural mass δ_1 for a simply-supported circular shear plate.

and, after some easy integrations, found to be:

$$M = (\delta_2/2) \times [1 + \sinh^2(ka)/k^2a^2] \tag{39}$$

As expected, the ratio is equal to zero when all the mass is structural ($\delta_1 = 1$), and tends to the value 1 when all the mass is nonstructural ($\delta_1 = 0$), and therefore not at the control of the designer.

A diametral cross section of the optimal plate corresponding to the case where 60% of the mass is nonstructural is represented in Fig. 9. The mass saving then obtained is equal to 8.6%. The variation of the optimal mass ratio with the proportion δ_1 of structural mass in the reference structure is plotted in Fig. 10: note that the savings obtained for a circular plate are slightly less important than those obtained in the case of a rectangular plate, as can be seen by comparison of the curves represented in Fig. 6 to the one of this figure.

One easily verifies[10] that classical one-dimensional optimization methods applied to this particular axisymmetrical problem yield the above solution. The much more challenging example of a plate with bending rigidity is now under study by similar methods, as a first step towards the minimum-mass design of a plate undergoing flutter for a given flutter speed.

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